In a nutshell: The adaptive Dormand-Prince method

Given the initial-value problem (IVP)

$$y^{(1)}(t) = f(t, y(t))$$
$$y(t_0) = y_0$$

we would like to approximate the solution $y(t)$ on the interval $[t_0, t_f]$ with a maximum error of ε_{abs} per unit time. This algorithm uses Taylor series and iteration. We start with an initial $h > 0$, we will have both minimum and maximum step sizes $h_{\min}$ and $h_{\max}$, respectively.

1. Let $k \leftarrow 0$.
2. If $t_k \geq t_f$, we are finished: we have approximated values for $y(t_1)$ through $y(t_k)$, and using cubic splines, we can approximate values at any point on the interval $[t_0, t_f]$.
3. If $k > N$, we will return signalling that too many steps were required to find the approximations.
4. Let
$$s_0 \leftarrow f(t_k, y_k)$$
$$s_1 \leftarrow f\left(t_k + \tfrac{1}{5}h, y_k + \tfrac{1}{5}hs_0\right)$$
$$s_2 \leftarrow f\left(t_k + \tfrac{3}{10}h, y_k + \tfrac{3}{10}h\frac{s_0 + 3s_1}{4}\right)$$
$$s_3 \leftarrow f\left(t_k + \tfrac{4}{5}h, y_k + \tfrac{4}{5}h\frac{11s_0 - 42s_1 + 40s_2}{9}\right)$$
$$s_4 \leftarrow f\left(t_k + \tfrac{8}{9}h, y_k + \tfrac{8}{9}h\frac{4843s_0 - 19020s_1 + 16112s_2 - 477s_3}{1458}\right)$$
$$s_5 \leftarrow f\left(t_k + h, y_k + h\frac{477901s_0 - 1806240s_1 + 1495424s_2 + 46746s_3 - 45927s_4}{167904}\right)$$
$$z \leftarrow y_k + h\frac{12985s_0 + 64000s_2 + 92750s_3 - 45927s_4 + 18656s_5}{142464}$$
$$s_6 \leftarrow f(t_k + h, z)$$
$$y \leftarrow y_k + h\frac{1921409s_0 + 9690880s_2 + 13122270s_3 - 5802111s_4 + 1902912s_5 + 534240s_6}{21369600}$$
each ratio is a weighted average, and y and z both approximate $y(t_k + h)$ but z is more accurate[1]
5. Let $a \leftarrow \sqrt[4]{\dfrac{h\varepsilon_{\text{abs}}}{2|y - z|}}$. *ah* estimates the ideal step size
6. If $a > 1$ or $h = h_{\min}$, we will set $t_{k+1} \leftarrow t_k + h$ and set $y_{k+1} \leftarrow z$ and then increment k.
If the ideal step size is greater than our current step size, or if the step size is already the minimum we will allow it, use z to approximate $y(t_k + h)$
7. If $0.9a < ½$, update $h \leftarrow ½h$,
if $0.9a > 2$, update $h \leftarrow 2h$,
otherwise update $h \leftarrow 0.9ah$. Update h with $0.9ah$ unless this more than doubles or halves its value
8. If $h < h_{\min}$, set $h \leftarrow h_{\min}$, and
if $h > h_{\max}$, set $h \leftarrow h_{\max}$. Don't let h exceed the lower or upper bounds we've set on it
9. Return to Step 2.

Note that Steps 1, 2, 3, and 6, 7, 8, 9 are identical to the adaptive Euler-Heun method, Step 4 only differs in how to find y and z, and Step 5 only differs by taking the 4th-root of the ratio.

[1] Normally, nutshells don't have such comments, but they are included here for clarity.

Important:

The coefficients in the calculations in Step 4 are written in the form

$$s_\ell \leftarrow f\left(t_k + c_\ell h, y_k + c_\ell h \frac{n_{\ell,0} s_0 + \cdots + n_{\ell,\ell-1} s_{\ell-1}}{d_\ell}\right).$$

This is so that you can clearly see that the ratio is a weighed average, as $n_{\ell,0} + \cdots + n_{\ell,\ell-1} = d_\ell$. When you are coding this, however, it is better to use the following:

$$s_\ell \leftarrow f\left(t_k + (c_\ell h), y_k + (c_\ell h)\left(\frac{n_{\ell,0}}{d_\ell} s_0 + \cdots + \frac{n_{\ell,\ell-1}}{d_\ell} s_{\ell-1}\right)\right).$$

This avoids multiplication by very large coefficients (for example, when multiplying the slopes by very large integers) and it ensures that we are not multiplying by very small numbers, for $c_\ell \le 1$ and *h* may be very small, which may occur if there is a discontinuity in *f*, which will occur if, for example, a switch is turned on or off. If you were to multiply the slopes by $\frac{c_\ell h n_{\ell,m}}{d_\ell} s_m$, if *h* is very small, it may make this a demormalized number, which will be calculating a sum of denormalized numbers, which will result in significant loss of precision. The formulation above, may still result in a denormalized number, but at least the sum will not be calculated at this increased loss of precision.

Acknowledgement:

A special thanks to Aristedes Jose B. Aquino Jr. who observed that one of the signs in one of the coefficients was incorrect.